



ELSEVIER

Contents lists available at SciVerse ScienceDirect

Journal of Symbolic Computation

journal homepage: www.elsevier.com/locate/jsc

Computational homology of n -types

Graham Ellis¹, Le Van Luyen

School of Mathematics, National University of Ireland, Galway, Ireland

ARTICLE INFO

Article history:

Received 6 July 2011

Accepted 2 February 2012

Available online 13 February 2012

Keywords:

Homology

 n -types

Crossed modules

Simplicial groups

ABSTRACT

We describe an algorithm for computing the integral homology of a simplicial group and illustrate an implementation on simplicial groups arising as the nerve of a category object in the category of groups.

© 2012 Published by Elsevier Ltd

1. Introduction

An n -type X is a connected CW-space with homotopy groups $\pi_i(X) = 0$ for all $i > n$. Up to homotopy equivalence such a space can be specified algebraically by means of a simplicial group G_* whose Moore complex is trivial in degrees greater than or equal to n . (All relevant simplicial definitions are recalled below.) More precisely, by treating each group G_i as a category with one object and constructing the nerve $N(G_i)_*$ one obtains a bisimplicial set $N(G_*)_*$. The diagonal $\Delta(G_*) = \{N(G_i)_i\}_{i \geq 0}$ is a simplicial set whose geometric realization is a CW-space $B(G_*)$. The condition on the Moore complex of G_* is sufficient to ensure that $B(G_*)$ is an n -type. The functor B induces an equivalence of categories

$$Ho(\text{Simplicial groups with Moore complex trivial in degrees } \geq n) \xrightarrow{\sim} Ho(n\text{-types})$$

where $Ho(C)$ denotes the category obtained from a category C by localizing with respect to those maps in C , termed *quasi-isomorphisms*, that induce isomorphisms on homotopy groups. We define the integral homology of a simplicial group G_* to be

$$H_i(G_*, \mathbb{Z}) = H_i(B(G_*), \mathbb{Z}), \quad i \geq 0.$$

On setting $n = 1$ this homology reduces to the classical homology theory of groups.

E-mail addresses: graham.ellis@nuigalway.ie (G. Ellis), v.le2@nuigalway.ie (L.V. Luyen).

¹ Tel.: +353 91493011.

A simplicial group G_* with Moore complex of length n is uniquely determined by its terms of degree less than or equal to n , and is thus easily represented on a computer. Our goal is to describe a practical algorithm that inputs this data and returns the corresponding integral homology groups. We need to impose a computability hypothesis on the input data. Let us say that a group G is k -computable if we have a practical algorithm for computing the first k terms of a free $\mathbb{Z}G$ -resolution R_*^G of \mathbb{Z} , with explicit contracting homotopy, in which each module R_j^G has finite rank. We deem G_* to be n -computable if its component groups G_i are $(n - i)$ -computable for $i \leq n$. In this article we describe an algorithm that inputs the first n terms of an n -computable simplicial group G_* together with an integer $i < n$, and outputs the abelian invariants of the group $H_i(G_*, \mathbb{Z})$.

An implementation of the algorithm (due to the second author) is available in the HAP package (Ellis, 2008) for the GAP computer algebra system (Ellis, 2006). We illustrate its performance on the homology of homotopy 2-types as there is particular interest in this special case. For instance, recent work of Baues and Bleile (2008) classifies, up to 2-torsion, homotopy classes of Poincaré duality complexes L in dimension 4 using *fundamental triples* comprising a homotopy 2-type X_L , a homomorphism $\omega: \pi_1(X_L) \rightarrow \{\pm 1\}$ and a homology class $H_4(X_L, \mathbb{Z}^\omega)$. Applications of the classification require calculations of the fourth homology of 2-types. Further interest in 2-types arises from basic constructions in group theory. Recall that a *crossed module* consists of a group homomorphism $\partial: M \rightarrow P$ and a group action $P \times M \rightarrow M$, $(p, m) \mapsto {}^p m$ satisfying

- (i) $\partial({}^p m) = p\partial(m)p^{-1}$
- (ii) $\partial m m' = m m' m^{-1}$

for $m, m' \in M$, $p \in P$. The homotopy groups of a crossed module are defined to be $\pi_1(\partial) = M/\partial(M)$, $\pi_2(\partial) = \ker(\partial)$, $\pi_i(\partial) = 0$ for $i > 2$. Three standard examples of a crossed module are as follows:

- (1) any normal subgroup M of a group P with inclusion ∂ and action given by conjugation.
- (2) the homomorphism $\partial: M \rightarrow \text{Aut}(M)$ from any group M to its automorphism group $P = \text{Aut}(M)$ which maps $m \in M$ to the inner automorphism $x \mapsto mxm^{-1}$.
- (3) the homomorphism $\partial: P \otimes P \rightarrow P$, $x \otimes y \mapsto xyx^{-1}y^{-1}$ from the tensor square of P (Brown et al., 1987). (If P is perfect then $P \otimes P$ is just the universal central extension. If P is abelian then $P \otimes P = P \otimes_{\mathbb{Z}} P$ is the usual tensor square of \mathbb{Z} -modules.)

For any crossed module $\partial: M \rightarrow P$ one can form the semi-direct product $M \rtimes P$ and the homomorphisms $s: M \rtimes P \rightarrow P$, $(m, p) \mapsto p$ and $t: M \rtimes P \rightarrow P$, $(m, p) \mapsto (\partial m)p$. One can regard this structure as a category C^∂ with $\text{Ob}(C^\partial) = P$ the set of objects and $\text{Arr}(C^\partial) = M \rtimes P$ the set of arrows, the source and target of arrows being determined by s and t . The category composition is given by $(m, p) \circ (m', p') = (mm', (\partial m')^{-1}p)$ whenever $p = (\partial m'p')$. The category composition and group composition are compatible in the sense that C^∂ is a category object in the category of groups. Such a category object is called a *cat¹-group*. The nerve $N(C^\partial)_*$ is a simplicial group. The Moore complex of $N(C^\partial)_*$ begins with the homomorphism $\partial: M \rightarrow P$ and is trivial in degrees > 1 . Thus $\pi_{i+1}(\partial) = \pi_i(N(C^\partial))$ for all $i \geq 0$. See for instance (Loday, 1982) for more information on crossed modules and their equivalence to: (i) cat¹-groups; (ii) simplicial groups with Moore complex of length 1; (iii) homotopy 2-types.

The integral homology of a crossed module ∂ was defined in Ellis (1992) to be the homology of the corresponding simplicial group

$$H_i(\partial: M \rightarrow P, \mathbb{Z}) = H_i(N(C^\partial)_*, \mathbb{Z}) \quad i \geq 0.$$

Theoretical aspects of this cohomology have been studied in several papers (Datuashvili and Pirashvili, 2001; Carrasco et al., 2002; Paoli, 2003). Ana Romero (Romero, 2007; Romero et al., 2009; Romero and Rubio, 0000) has written Lisp code for computing the integral homology of a crossed module. Her method uses the fibration sequence

$$K(\pi_2(X), 2) \rightarrow X \rightarrow K(\pi_1(X), 1)$$

associated to any homotopy 2-type X and the classical homological perturbation lemma (Brown, 1967) to obtain a small algebraic model for X in terms of small models for the Eilenberg Mac Lane spaces

$K(\pi_2(X), 2)$ and $K(\pi_1(X), 1)$. Her code is available as a module for the Kenzo system (Sergeraert, 0000) for computations in Algebraic Topology.

The present paper is motivated by the work of Romero and her collaborators. We also use (a generalized version of) the homological perturbation lemma. A difference is that, instead of constructing a small simplicial model for the total space of a fibration, we aim for a small chain complex $K(G_*)$ that is chain equivalent to the chain complex of the diagonal simplicial set $\Delta(G_*)$ of an arbitrary n -computable simplicial group G_* . A second difference is that we need no assumption about the action of the fundamental group on higher homotopy groups.

The following GAP session illustrates how our implementation can be used to compute

$$H_4(\partial: G \rightarrow \text{Aut}(G), \mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

where G is the dihedral group of order 32. The cat^1 -group C^∂ has underlying group of order 4096 and homotopy groups $\pi_1(\partial) = C_4 \times C_2$, $\pi_2(\partial) = \mathbb{Z}_2$. To reduce computations a quasi-isomorphism of cat^1 -groups $C^\partial \simeq D^\partial$ is constructed. The underlying group of D^∂ is non-abelian of order 64.

```
gap> C:=AutomorphismGroupAsCatOneGroup(DihedralGroup(32));;
gap> StructureDescription(HomotopyGroup(C,1));
"C4 x C2"
gap> StructureDescription(HomotopyGroup(C,2));
"C2"
gap> D:=QuasiIsomorph(C);;
gap> N:=NerveOfCatOneGroup(D,5);;
gap> K:=ChainComplexOfSimplicialGroup(N);;
gap> Homology(K,4);
[ 2, 2, 2 ]
```

These commands took 50 s on a Linux 1.8 GHz laptop with 1GB ram. We are not aware of any software that could readily be used to benchmark the computation (although in principle the code of Romero et al. (2009) could be used to benchmark part of the computation since, in this example, $\pi_1(\partial)$ acts trivially on $\pi_2(\partial)$). Correctness of the computation can be partially tested by using the nerve of C^∂ , rather than the nerve of D^∂ , to obtain the same homology result. Analogous commands could be used to compute the homology of Eilenberg–Mac Lane spaces $K(\pi, n)$ for all $n \geq 1$. However, for $n = 1, 2, 3$ there exist far more efficient computational approaches to these spaces: the HAP package (Ellis, 2008) provides a range of implemented algorithms for the case $n = 1$; the thesis of Clément (2002) provides an extremely efficient implementation of a technique of Cartan for $n = 2, 3$.

2. Representing resolutions of groups

We say that a free $\mathbb{Z}G$ -resolution R_*^G of \mathbb{Z} is *finitely generated* if the free $\mathbb{Z}G$ -module R_i^G has finite rank for each degree $i \geq 0$. The first n terms of such a resolution can be represented on a computer by storing:

- the $\mathbb{Z}G$ -rank of the i th free module R_i^G ($i \leq n$).
- the image of the k th free $\mathbb{Z}G$ -generator of R_i^G under the boundary homomorphism $d_i: R_i^G \rightarrow R_{i-1}^G$ ($i \leq n$).
- the image of the k th free \mathbb{Z} -generator of R_i^G under a contracting homotopy $h_i: R_i^G \rightarrow R_{i+1}^G$ ($0 \leq i \leq n-1$).

The contracting homotopy satisfies, by definition, $h_{i-1}d_i + d_{i+1}h_i = 1$ and needs to be specified on a set of free \mathbb{Z} -module generators of R_i^G since it is not G -equivariant. The homotopy can be used to make algorithmic the following frequent element of choice.

For $x \in \ker(d_i: R_i^G \rightarrow R_{i-1}^G)$ choose an element $\tilde{x} \in R_{i+1}^G$ such that $d_{i+1}(\tilde{x}) = x$.

One sets $\tilde{x} = h_i(x)$. In particular, for any group homomorphism $\phi: G \rightarrow G'$, the contracting homotopy on a free $\mathbb{Z}G'$ -resolution $R_*^{G'}$ provides an explicit induced ϕ -equivariant chain map $\phi_*: R_*^G \rightarrow R_*^{G'}$.

The HAP package (Ellis, 2008) provides implementations of practical algorithms for computing the first n terms of finitely generated free $\mathbb{Z}G$ -resolutions for: finite groups (Ellis, 2004; Ellis et al., 2006; Dutour Sikric and Ellis, 2009; Green, 2003), finitely generated nilpotent groups (Wall, 1961), crystallographic groups (Ellis, 2008; Röder, 2010), various arithmetic groups (Ellis and Williams, 2005; Dutour Sikric et al., 2011), Coxeter groups (Ellis et al., 2006) and certain Artin groups (Ellis and Sköldberg, 2010). In this context “practical” implies that for reasonably small values of $n \geq 0$ the rank of R_n^G is reasonably small.

The assignment $G \mapsto R_*^G$ of an arbitrary free $\mathbb{Z}G$ -resolution to each group G is clearly not functorial. The assignment can be made functorial by associating the *bar resolution* B_*^G to each group. Recall that B_n^G is the free $\mathbb{Z}G$ -module generated by n -tuples $[g_1|g_2|\cdots|g_n]$ ($g_i \in G$). The boundary homomorphism is $\partial = \sum_{i=0}^n (-1)^i d_i: B_n^G \rightarrow B_{n-1}^G$ where

$$d_i[g_1|\cdots|g_n] = \begin{cases} g_1[g_2|\cdots|g_n] & i = 0 \\ [g_1|\cdots|g_i g_{i+1}|\cdots|g_n] & 0 < i < n \\ [g_1|\cdots|g_{n-1}] & i = n. \end{cases} \quad (1)$$

A contracting homotopy $h_n: B_n^G \rightarrow B_{n+1}^G$ is given by

$$g[g_1|\cdots|g_n] \mapsto [g|g_1|\cdots|g_n].$$

We define the *bar complex* \bar{B}_*^G to be the chain complex of free abelian groups given by

$$\bar{B}_*^G = B_*^G \otimes_{\mathbb{Z}G} \mathbb{Z}$$

where \mathbb{Z} has trivial G -action. The bar complex is also a functorial construction.

Note that the module B_n^G is not of finite rank when G is infinite. Even for finite groups the rank of B_n^G is large and it is typically not practical to compute the homology of G from its bar complex by directly applying the Smith Normal Form algorithm. However, for both finite and infinite groups it is straightforward to represent an arbitrary word $w \in B_n^G$ (i.e. a $\mathbb{Z}G$ -linear combination of generators) on a computer, together with the formulae for $d_n(w)$ and $h_n(w)$. We deem such representations of w , $d_n(w)$, $h_n(w)$ to constitute a computer implementation of the bar resolution B_*^G .

Given computer implementations of a finitely generated $\mathbb{Z}G$ -resolution R_*^G of \mathbb{Z} , and of the bar resolution B_*^G , it is straightforward to compute $\mathbb{Z}G$ -equivariant chain equivalences $\iota: R_*^G \rightarrow B_*^G$ and $\psi: B_*^G \rightarrow R_*^G$. These two chain maps can be represented as functions that return the images $\iota(v)$, $\psi(w)$ of arbitrary words $v \in R_n^G$, $w \in B_n^G$. It is also straightforward to compute $\mathbb{Z}G$ -equivariant homomorphisms $H_n: B_n^G \rightarrow B_{n+1}^G$ ($n \geq 0$) that constitute a chain homotopy $H: \iota\psi \simeq 1$.

Definition. We call the tuple $(R_*^G, B_*^G, \iota, \psi, H)$ an *effective bar resolution*.

The notion of an effective bar resolution is a mechanism for algorithmically blending the functorial properties of the bar resolution with the computational advantages of a small finitely generated resolution. An effective bar resolution induces chain equivalences $\iota: \bar{R}_*^G \rightarrow \bar{B}_*^G$, $\psi: \bar{B}_*^G \rightarrow \bar{R}_*^G$ and chain homotopy $H: \iota\psi \simeq 1$ where $\bar{R}_*^G = R_*^G \otimes_{\mathbb{Z}G} \mathbb{Z}$.

Definition. We call the induced tuple $(\bar{R}_*^G, \bar{B}_*^G, \iota, \psi, H)$ an *effective bar complex*.

An effective bar complex is almost an example of a *chain complex with effective homology* as used in Rubio and Sergeraert’s theory of *effective homology* (Rubio and Sergeraert, 2005, 2006; Sergeraert, 0000). However, for computational reasons we drop their requirement that R_*^G be a strong deformation retract of B_*^G . We remark that Rubio and Sergeraert’s theory has been applied by Romero (Romero, 2007; Romero et al., 2009; Romero and Rubio, 0000) to the computation of the homology of homotopy 2-types. The application of effective bar complexes given below differs slightly from Romero’s work in that: (i) it applies to homotopy n -types, and (ii) it does not explicitly use the twisted cartesian product of a fibration.

3. Computing small chain complexes for simplicial groups

Recall that a *simplicial group* G_* consists of an infinitely countable sequence of groups $G_n (n \geq 0)$ and homomorphisms $d_i^n: G_n \rightarrow G_{n-1} (n \geq 1, 0 \leq i \leq n)$ satisfying certain *simplicial identities*. The identities are succinctly expressed by saying that the simplicial group is a functor $\Delta^{op} \rightarrow (\text{Groups})$ where Δ is the category whose objects are the sets $[n] = \{0, 1, \dots, n\} (n \geq 0)$ and whose morphisms are nondecreasing maps. The *Moore complex* MG_* of a simplicial group is the chain complex of nonabelian groups $MG_n = \cap_{1 \leq i \leq n} \ker d_i^n$, $MG_0 = G_0$ with boundary homomorphisms $d_0^n: MG_n \rightarrow MG_{n-1}$, $d_0^0 = 0$. The *homotopy groups* of a simplicial group are defined to be the homology groups of its Moore complex,

$$\pi_n(G_*) = \ker(d_0^n: MG_n \rightarrow MG_{n-1}) / \text{Image}(d_0^{n+1}: MG_{n+1} \rightarrow MG_n)$$

for $n \geq 0$.

If the groups G_n are abelian for $n \geq 0$ then there is a second chain complex AG_* associated to G_* . It has $AG_n = G_n$ for $n \geq 0$ with boundary homomorphism given by $\partial_n = \sum_{i=0}^n (-1)^i d_i^n: AG_n \rightarrow AG_{n-1}$. We shall call AG_* the *alternating chain complex*. The simplicial identities imply isomorphisms $\pi_n(G_*) \cong H_n(AG_*)$.

In general one defines a simplicial object in a category C to be a functor $\Delta^{op} \rightarrow C$. For short we use terms such as *simplicial sets* for C the category of sets and *bisimplicial sets* for C the category of simplicial sets. We use the term *bicomplex* for a chain complex in the category of chain complexes of free abelian groups.

By applying the bar complex construction to the terms in a simplicial group G_* we obtain a chain complex $\bar{B}_*^{G_*}$ of simplicial abelian groups. Taking the alternating chain complex of each simplicial abelian group $\bar{B}_i^{G_*}$ yields a bicomplex which we denote by $\bar{AB}_*^{G_*}$.

Another way to construct this bicomplex is via the functor $F: (\text{Sets}) \rightarrow (\text{Abelian groups})$ that sends a set to the free abelian group generated by the set. By applying F to the bisimplicial set $N(G_*)_*$ we get a bisimplicial abelian group $FN(G_*)_*$. Then replacing each *horizontal* simplicial abelian group $FN(G_*)_n$, and each *vertical* simplicial abelian group $FN(G_n)_*$, by the associated alternating chain complex we obtain a bicomplex $AFN(G_*)_*$. The bicomplexes $\bar{AB}_*^{G_*}$ and $AFN(G_*)_*$ are isomorphic. The Eilenberg–Zilber Theorem (see Weibel, 1994) implies that the diagonal chain complex of $AFN(G_*)_*$ is chain homotopic to the total chain complex of $AFN(G_*)_*$. We thus obtain the following alternative description of the integral homology of a simplicial group G_* .

$$H_n(G_*, \mathbb{Z}) = H_n(\text{Tot}(\bar{AB}_*^{G_*})), \quad n \geq 0. \quad (2)$$

Recall that the total chain complex has $\text{Tot}(\bar{AB}_*^{G_*})_n = \oplus_{p+q=n} \bar{AB}_p^{G_q}$ and boundary homomorphism $\partial(x) = d^v(x) + \delta^h(x)$ for $x \in \bar{AB}_p^{G_q}$ where d^v, δ^h are the vertical and horizontal boundaries in the bicomplex $\bar{AB}_*^{G_*}$ and $\delta^h(x) = (-1)^p d^h(x)$.

The following theorem describes our basic algorithm for computing the homology of a simplicial group. It provides easily implemented formulae for computing a small chain complex K_* from which the desired homology can be extracted by a direct application of the Smith Normal Form algorithm.

Theorem 1. Suppose that G_* is a simplicial group and that we have an effective bar complex $(\bar{R}_*^{G_p}, \bar{B}_*^{G_p}, \iota, \psi, H)$ for each $p \geq 0$. Then the total complex $\text{Tot}(\bar{AB}_*^{G_*})$ is chain homotopic to a chain complex (K_*, δ') with

$$K_n = \oplus_{p+q=n} \bar{R}_p^{G_q}$$

and with boundary homomorphism

$$\delta' = d^v + \psi \delta^h \iota + \psi \delta^h H \delta^h \iota + \psi \delta^h H \delta^h H \delta^h \iota \dots$$

Theorem 1 is an immediate consequence of a generalization, due to [Crainic \(2004\)](#), of the classical homological perturbation lemma ([Brown, 1967](#)). This generalization is stated using the notion of a *homotopy equivalence data*

$$(L, d) \xrightarrow[\iota]{\psi} (M, d), H \quad (3)$$

which comprises chain complexes L, M , quasi-isomorphisms ι, ψ and a homotopy $\iota\psi - 1 = dH + Hd$. A *perturbation* on (3) is a homomorphism $\delta: M \rightarrow M$ of degree -1 such that $(d + \delta)^2 = 0$.

Lemma 2 (*Generalized Homological Perturbation Lemma* ([Crainic, 2004](#))). Suppose given a perturbation δ on a homotopy equivalence data (3) for which $(1 - \delta H)^{-1}$ exists. Set $A = (1 - \delta H)^{-1}\delta$. Then

$$(L, d') \xrightarrow[\iota']{\psi'} (M, d + \delta), H' \quad (4)$$

is a homotopy equivalence data where

$$\iota' = \iota + HA\iota, \quad \psi' = \psi + \psi AH, \quad H' = H + HAH, \quad d' = d + \psi A\iota.$$

We refer the reader to [Crainic \(2004\)](#) for a proof of **Lemma 2**. **Theorem 1** is obtained from the lemma by taking (M, d) to be the chain complex $(\oplus_{p+q=n} \bar{A}\bar{B}_p^{Gq}, d^v)$ and taking δ to be the perturbation δ^v .

The classical perturbation lemma requires the homotopy equivalence data (3) to satisfy the extra requirements

$$\psi\iota = 1, \quad \iota H = 0, \quad H^2 = 0. \quad (5)$$

A version of **Theorem 1** derived from the classical lemma can be found in [Rubio and Sergeraert \(2006\)](#). This classical version is less practical for our purposes as the extra requirement $\psi\iota$ on an effective bar complex would necessitate extra computations.

4. Nerves and quasi-isomorphisms of crossed modules

The *nerve* of a small category $s, t: C \rightarrow \text{Ob}(C)$ is a simplicial set NC_* with $NC_0 = \text{Ob}(C)$ and $NC_k = \{(x_1, \dots, x_k): x_i \in C, t(x_i) = s(x_{i+1})\}$ for $k \geq 1$. The first boundary homomorphisms $d_i: NC_1 \rightarrow NC_0$ are $d_0 = s, d_1 = t$. For $k \geq 2$ the boundary homomorphisms $d_i: NC_k \rightarrow NC_{k-1}$ are given by composition at the i th object: $d_0(x_1, \dots, x_k) = (x_2, \dots, x_k)$, $d_i(x_1, \dots, x_k) = (x_1, \dots, x_i x_{i+1}, \dots, x_k)$ for $1 \leq i \leq k-1$, $d_k(x_1, \dots, x_k) = (x_1, \dots, x_{k-1})$.

If C is a cat^1 -group then the nerve is naturally a simplicial group. In this case it is readily checked that the Moore complex of the nerve is trivial in degrees greater than 1; in degrees 0 and 1 it coincides with the crossed module associated to C .

A *morphism* between two cat^1 -groups $s, t: C \rightarrow \text{Ob}(C), s', t': C' \rightarrow \text{Ob}(C')$ is a group homomorphism $\phi: C \rightarrow C'$ satisfying $\phi s = s' \phi, \phi t = t' \phi$. We say that ϕ is a *quasi-isomorphism* if it induces isomorphisms on the homotopy groups $\pi_1(C) = \text{Ob}(C)/t(\ker s), \pi_2(C) = \ker s \cap \ker t$ of the associated crossed module.

Quasi-isomorphic cat^1 -groups have quasi-isomorphic nerves and isomorphic homology. A reasonable first step towards computing homology of a cat^1 -group is thus to try to find a computationally easier quasi-isomorphic copy. In the case of a finite cat^1 -group this essentially means searching for a quasi-isomorphic copy whose underlying group has lower order; we can do this as follows.

Procedure 3. (i) Given a finite cat^1 -group $s, t: C \rightarrow \text{Ob}(C)$ we can search through the normal subgroups of $\ker s$ to find a largest normal subgroup K in $\ker s$ such that K is normal in G and $K \cap \ker t = 1$. We can then choose some generating set X_K for K and construct the group N generated by the set $X_K \cup \{t(x) : x \in X_K\}$. Then N is a normal subgroup of C for which the quotient homomorphism $C \rightarrow C/N$ is a quasi-isomorphism. (ii) Given a finite cat^1 -group $s, t: C \rightarrow \text{Ob}(C)$ we can search through the subgroups of C to find a smallest cat^1 -group $K \subset C$ such that the inclusion $K \hookrightarrow C$ is a quasi-isomorphism. (iii) We can repeatedly apply (i) followed by (ii).

The assertion that $C \rightarrow C/N$ is a quasi-isomorphism in **Procedure 3**(i) is readily checked.

For the dihedral group G of order 32 the crossed module $\partial: G \rightarrow \text{Aut}(G)$ is equivalent to a cat^1 -group C^∂ whose underlying group has order 4096. By the above method we can construct a subgroup $H^\partial \leq C^\partial$ of order 64 that gives rise to a quasi-isomorphism $H^\partial \rightarrow C^\partial$. The orders of the groups in the nerves of C^∂ and H^∂ are related by

$$|N(C^\partial)_k| = 8^{k+1} |N(H^\partial)_k|.$$

5. An idea for improving the homology computations

The chain complex constructed for a simplicial group using [Theorem 1](#) and [Procedure 3](#) is typically unnecessarily large. Given any chain complex (A_*, d) of finitely generated free abelian groups there are a number of ways in which one might attempt to produce a chain homotopy equivalence $A_* \simeq B_*$ where (B_*, d') is a chain complex of free abelian groups of lower ranks. We shall describe one such procedure which grew out of conversations with Pawel Dlotko, T. Kaczynski and Marian Mrozek and is based on ideas in their paper ([Dlotko et al., 2011](#)). It is extremely simple to implement yet surprisingly effective in many cases.

Let us denote by e_i^n the free generators of A_n . Let us define a pair (e_i^n, e_j^{n-1}) to be *redundant* if $d(e_i^n) = \pm e_j^{n-1}$. A redundant pair generates a sub chain complex $\langle e_i^n, e_j^{n-1} \rangle$ of A_* with trivial homology. The long exact homology sequence arising from a short exact sequence of chain complexes implies that the quotient chain map $\Pi_*: A_* \rightarrow A_*/\langle e_i^n, e_j^{n-1} \rangle$ induces isomorphisms on homology. Moreover, the quotient $A_*/\langle e_i^n, e_j^{n-1} \rangle$ is a chain complex of free abelian groups and hence Π_* must be a chain homotopy equivalence.

Procedure 4. Given a finite dimensional chain complex A_* we set $A_*^{(0)} = A_*$ and, while $A_*^{(k)}$ contains a redundant pair (e_i^n, e_j^{n-1}) , we set $A_*^{(k+1)} = A_*^{(k)} / \langle e_i^n, e_j^{n-1} \rangle$. When $A_*^{(k)}$ contains no redundant pair we set $B_* = A_*^{(k)}$. The composite of chain homotopy equivalences

$$A_*^{(0)} \rightarrow A_*^{(1)} \rightarrow \dots \rightarrow A_*^{(k)}$$

provides a chain homotopy equivalence $A_* \rightarrow B_*$.

An implementation of [Procedure 4](#) is available in HAP ([Ellis, 2008](#)). To illustrate its performance we consider the crossed module $\partial: \mathbb{Z}_2 \rightarrow 0$ representing an Eilenberg–Mac Lane space $K(\mathbb{Z}_2, 2)$. Let $K_* = C_*(N(C^\partial))$ denote the chain complex for the nerve of the cat^1 -group C^∂ constructed using [Theorem 1](#). When [Procedure 4](#) is applied to this chain complex K_* it yields a chain homotopy equivalence $K_* \simeq L_*$ where the ranks of K_i and L_i are listed in the following table for low degrees.

i	0	1	2	3	4	5	6	7	8	9	10
$\text{Rank}(K_i)$	1	1	2	4	8	16	32	64	128	256	512
$\text{Rank}(L_i)$	1	0	1	1	2	3	5	8	13	21	34

Experimental evidence seems to suggest that [Procedure 4](#) yields a free abelian chain complex for $K(\mathbb{Z}_m, 2)$, $m \geq 2$, whose terms have ranks equal to the Fibonacci numbers. We remark that [Berger \(2007\)](#) has proved the existence of a CW-complex of type $K(\mathbb{Z}_2, 2)$ whose terms have ranks equal to the Fibonacci numbers.

We now come to the idea for improving our procedure for computing homology of n -types. Suppose that

$$1 \rightarrow N_* \rightarrow G_* \rightarrow Q_* \rightarrow 1 \quad (6)$$

is a short exact sequence (i.e. *fibration sequence*) of simplicial groups. Using the perturbation technique explained in Wall's paper ([Wall, 1961](#)) the HAP package ([Ellis, 2008](#)) can produce effective bar complexes $(\bar{R}_*^{G_p}, \bar{B}_*^{G_p}, \iota, \psi, H)$ in which each chain complex $\bar{R}_*^{G_p}$ has the form of a *twisted tensor product* $\bar{R}_*^{G_p} = \bar{R}_*^{N_p} \bar{\otimes} \bar{R}_*^{Q_p}$. The free abelian group underlying $\bar{R}_*^{N_p} \bar{\otimes} \bar{R}_*^{Q_p}$ is the same as that underlying the usual tensor product of chain complexes $\bar{R}_*^{N_p} \otimes \bar{R}_*^{Q_p}$; the differential on the twisted tensor product

is a perturbation of the usual differential. The chain complex KG_* constructed for the simplicial group G_* using [Theorem 1](#) thus has the form of a twisted tensor product

$$KG_* = KN_* \tilde{\otimes} KQ_*$$

(whose underlying free abelian group coincides with that of $KG_* = KN_* \otimes KQ_*$).

Suppose now that for the chain complexes KN_* and KQ_* produced using [Theorem 1](#) we have used [Procedure 4](#) to obtain homotopy equivalence data

$$(LN_*, d) \xrightarrow[\psi]{\ell} (KN_*, d), H, \quad (7)$$

$$(LQ_*, d) \xrightarrow[\psi]{\ell} (KQ_*, d), H. \quad (8)$$

We can combine (7) and (8) to produce a homotopy equivalence data

$$(LN_* \otimes LQ_*, d) \xrightarrow[\psi]{\ell} (KN_* \otimes KQ_*, d), H. \quad (9)$$

[Lemma 2](#) can be applied to a perturbation on the right hand side of (9) to obtain a homotopy equivalence data

$$(LN_* \tilde{\otimes} LQ_*, d) \xrightarrow[\psi]{\ell} (KN_* \tilde{\otimes} KQ_*, d), H. \quad (10)$$

The left hand side of (10) is a small chain complex for computing the homology of G_* . Note that by using a lazy evaluation programming style, an implementation of the chain complex $LN_* \tilde{\otimes} LQ_*$ does not require all differentials in KG_* to be computed.

We have only implemented this improvement for the easy case where $G_* = N_* \times Q_*$ in the fibration sequence (6). As an illustration let us take $G = C_3 \rtimes C_8$ to be the first group of order 24 in the GAP small groups library. The crossed module $\partial: G \rightarrow \text{Aut}(G)$ corresponds to a cat^1 -group C^∂ of order 576. There is a quasi-isomorphism $C^\partial \simeq D^\partial$ with D^∂ a cat^1 -group of order 16. Now $\pi_1(\partial) = C_2 \times C_2$ and $\pi_2(\partial) = C_4$. By construction $\pi_1(\partial)$ acts trivially on $\pi_2(\partial)$. It follows that C^∂ represents the homotopy type $K(C_4, 2) \times K(C_2 \times C_2, 1)$. Using [Theorem 1](#) and [Procedure 4](#) we can compute a chain complex LN_* for $K(C_4, 2)$. We can compute a minimal chain complex LQ_* for $K(C_2 \times C_2, 1)$. The tensor product $LN_* \otimes LQ_*$ yields the homology of the crossed module ∂ . Ranks for these chain complexes in low degrees are given in the following table.

i	0	1	2	3	4	5	6
$\text{Rank}(LN_i)$	1	0	1	3	8	19	46
$\text{Rank}(LQ_i)$	1	2	3	4	5	6	7
$\text{Rank}(LN_* \otimes LQ_*)_i$	1	2	4	9	22	54	132

We have $H_5(C^\partial, \mathbb{Z}) = H_5(LN_* \otimes LQ_*) = (\mathbb{Z}_2)^{11}$.

Acknowledgements

The second author has been supported by an NUI Galway, College of Science, Ph.D. fellowship.

References

- Baues, H.-J., Bleile, B., 2008. Poincaré duality complexes in dimension four. *Algebr. Geom. Topol.* 8, 2355–2389.
- Berger, C., 2007. Iterated wreath product of the simplex category and iterated loop spaces. *Adv. Math.* 213, 230–270.
- Brown, R., 1967. The twisted Eilenberg–Zilber theorem. *Celebrazioni Arch. Secolo XX*, 34–37. *Simp. Top.*
- Brown, R., Johnson, D.L., Robertson, E.F., 1987. Some computations of non-abelian tensor products of groups. *J. Algebra* 111, 177–202.
- Carrasco, P., Cegarra, A.M., Grandjen, A.R., 2002. (Co)homology of crossed modules. In: *Category theory 1999 (Coimbra)*. *J. Pure Appl. Algebra* 168 (2–3), 147–176.

- Clément, A., 2002. Integral cohomology of finite Postnikov towers. Thèse de doctorat, Université de Lausanne.
- Crainic, M., 2004. On the perturbation lemma, and deformations, p. 13. [arXiv:math/0403266v1](https://arxiv.org/abs/math/0403266v1).
- Datuashvili, T., Pirashvili, T., 2001. On (co)homology of 2-types and crossed modules. *J. Algebra* 244 (1), 352–365.
- Dlotko, P., Kaczynski, T., Mrozek, M., Wanner, T., 2011. Coreduction homology algorithm for regular CW-complexes. *Discrete Comput. Geom.* 46 (2), 361–388.
- Dutour Sikric, M., Ellis, G., 2009. Wythoff polytopes and low-dimensional homology of Mathieu groups. *J. Algebra* 322 (11), 4143–4150.
- Dutour Sikric, M., Ellis, G., Schürmann, A., 2011. The integral cohomology of $PSL_4(\mathbb{Z})$ and other arithmetic groups. *J. Number Theory* 131 (12), 2368–2375.
- Ellis, G., 1992. Homology of 2-types. *J. Lond. Math. Soc.* (2) 46 (1), 1–27.
- Ellis, G., 2008. HAP – Homological Algebra programming, Version 1.8, an official package for the GAP computational algebra system. (<http://www.gap-system.org/Packages/hap.html>).
- Ellis, G., 2004. Computing group resolutions. *J. Symbolic Comput.* 38 (3), 1077–1118.
- Ellis, G., Harris, J., Sköldberg, E., 2006. Polytopal resolutions for finite groups. *J. Reine Angew. Math.* 598, 131–137.
- Ellis, G., Sköldberg, E., 2010. The $K(\pi, 1)$ conjecture for a class of Artin groups. *Comment. Math. Helv.* 85 (2), 409–415.
- Ellis, G., Williams, A.G., 2005. On the cohomology of generalized triangle groups. *Comment. Math. Helv.* 80 (3), 571–591.
- The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.4.9. (<http://www.gap-system.org>).
- Green, D.J., 2003. Gröbner bases and the computation of group cohomology. In: *Lecture Notes in Math.*, vol. 1828. Springer-Verlag, Berlin.
- Loday, J.-L., 1982. Spaces with finitely many non-trivial homotopy groups. *J. Pure Appl. Algebra* 24 (2), 179–202.
- Paoli, S., 2003. (Co)homology of crossed modules with coefficients in a π_1 -module. *Homology, Homotopy Appl.* 5 (1), 261–296.
- Röder, M., 2010. Geometric algorithms for resolutions for Bieberbach groups. In: *Computational group theory and the theory of groups, II. Contemp. Math.* 511, 167–178.
- Romero, A., 2007. Homología efectiva y sucesiones espectrales, Ph.D. thesis, Univ. de la Rioja.
- Romero, A., Ellis, G., Rubio, J., 2009. Interoperating between computer algebra systems: computing homology of groups with Kenzo and GAP, ISAAC Proceedings, pp. 303–310.
- Romero, A., Rubio, J., Computing the homology of groups: the geometric way. *J. Symbolic Comput.* doi:10.1016/j.jsc.2011.12.007.
- Rubio, J., Sergeraert, F., 2005. Algebraic models for homotopy types. *Homology, Homotopy Appl.* 7 (2), 139–160.
- Rubio, J., Sergeraert, F., 2006. Constructive homological algebra and applications, summer school lecture notes, Genova.
- Sergeraert, F., Kenzo software system for computations in algebraic topology, <http://www-fourier.ujf-grenoble.fr/~sergerar/Kenzo>.
- Wall, C.T.C., 1961. Resolutions of extensions of groups. *Proc. Cambridge Philos. Soc.* 57, 251–255.
- Weibel, C., 1994. An Introduction to Homological Algebra. In: *Cambridge Studies in Advanced Mathematics*, vol. 38. Cambridge, p. xiv+450.